
BI-MULTIFRACTAL ANALYSIS AND MULTI-AFFINE MODELING OF NON-STATIONARY GEOPHYSICAL PROCESSES, APPLICATION TO TURBULENCE AND CLOUDS

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Abstract

In recent years, considerable progress has been achieved in the description of natural variability, largely due to the widespread use of scale-invariant concepts such as fractals and multifractals. In particular, this last concept has been used to clarify the fuzzy notion of "inhomogeneity" by introducing and quantifying the effects of intermittency. In this paper, we present a more comprehensive approach to multifractal data analysis and simulation that includes and combines the currently popular singularity analysis techniques with the more traditional approach based on structure functions. Being related to the new idea of "multi-affinity", these last statistics are regaining favor and constitute the proper framework to address the problem of quantifying and qualifying yet another outstanding fuzzy notion, that of "non-stationarity". This is an important step because non-stationary behavior is ubiquitous in Nature.

Using turbulence as an example, we also show how a unified multifractal formalism can help in extracting, from data alone, the "effective constitutive laws" that describe phenomenologically the nonlinearities of the macroscopic transport processes that shape the geophysical field represented by the dataset. Finally, we argue that the essential multifractality of any natural system can be captured on the " $q = 1$ multifractal plane" and describe ways in which it can be used in practical geophysical problems.

1. MOTIVATION AND OVERVIEW

Most data analysis techniques in usage (including the currently popular multifractal ones), make at least implicitly stationarity assumptions about the data, viewed as a representative realization of a stochastic process. Nature in the meantime is providing us with ample evidence of non-stationary behavior. In the following section, we show that “multi-affinity”¹ is the proper framework when dealing with non-stationary scale-invariant processes since directly related to structure functions. In Sec. 3 we survey multi-singularity analysis as a means to characterize quantitatively and qualitatively intermittency. The possibilities opened simply by connecting the two multi-scaling approaches are very promising (Sec. 4). Finally, we synthesize our outlook with the help of the “ $q = 1$ multifractal plane”.

2. MULTI-AFFINITY ANALYSIS, QUANTIFYING AND QUALIFYING NON-STATIONARITY

2.1 Scale-invariance, Stationarity and Stationary Increments

We take $\phi(x)$ to be a generic geophysical signal representative of either a time-series or a field assumed, for simplicity, to be a one-dimensional probing of some kind. More specifically, we are given $N + 1$ real values over the interval $[0, L]$, sampled at rate $1/\ell$:

$$\phi(x), \quad \frac{x}{\ell} = 0, 1, \dots, N \left(N = \frac{L}{\ell} \right). \quad (1)$$

In a typical situation, the above signal is highly fluctuating, “rough”, possibly discontinuous. This is an indication that complicated, strongly nonlinear physical processes are at work generating $\phi(x)$, and the only symmetry we can reasonably posit for the system is scale-invariance. We will therefore use scale, denoted r , as a parameter in all of our statistics and seek statistically robust power-law behavior with respect to r . In sharp contrast with mathematical models, real world systems only scale over a finite range going from the “inner” (or “homogeneity” or “dissipation”) scale η_ϕ to the “outer” (or “forcing” or “integral”) scale R_ϕ . The instrumentally accessible portion of this range is $[\max\{\ell, \eta_\phi\}, \min\{L, R_\phi\}]$ but, for simplicity, we will take $\ell = \eta_\phi$ and $L = R_\phi$ throughout the following.

Our first task is to define “stationary” features of the dataset $\phi(x)$ to work on, with statistical stationarity meaning in essence “invariance under translation”. The simplest scale-dependent statistic of a stationary process is its 2-point auto-correlation function:

$$G_\phi(r) = \langle \phi(x)\phi(x-r) \rangle \sim r^{-\mu_\phi} (\ell \leq r \leq L), \quad 0 < \mu_\phi < 1 \quad (2a)$$

where $\langle \cdot \rangle$ designates ensemble-averaging. The exponent μ_ϕ has a natural range: correlations are generally expected to decrease with increasing separation ($\mu_\phi > 0$) but too rapid decays ($\mu_\phi \geq 1$) are basically equivalent to “ δ -correlations”.

An even more popular 2nd order statistic is the energy spectrum $E_\phi(k)$:

$$E_\phi(k) = \sum_{\pm} \langle \hat{\phi}(\pm k) \hat{\phi}(\pm k)^* \rangle \sim k^{-\beta_\phi} \left(\frac{1}{L} \leq k \leq \frac{1}{2\ell} \right), \quad 0 < \beta_\phi = 1 - \mu_\phi < 1 \quad (2b)$$

where $\hat{\cdot}$ designates a Fourier transform. The two statistics are closely related since $G_\phi(r)$ and $E_\phi(k)$ are in Fourier duality (Wiener-Khinchine theorem). In scale-invariant situations,

this leads to the connection between the two exponents in Eqs. (2). We are more interested in the limit, $\mu_0 \rightarrow 0$ ($\beta_\phi \rightarrow 1$), where we anticipate long-range correlations which are to be interpreted as a symptom of non-stationarity. We will focus more specifically on processes with $1 < \beta_\phi < 3$ which are said to have “stationary increments”. Many turbulent^{2–4} or otherwise natural^{5,6} signals are in this class.

In the up-coming subsection, we characterize the multi-scaling properties of increments themselves over the full range of available scales, thus dealing with the non-stationarity directly. In Sec. 3, we adopt an altogether different strategy by first deriving from $\phi(x)$ another field which is itself stationary and can justifiably be investigated using methods requiring this property. This new field is required to contain the most interesting information on $\phi(x)$, specifically concerning its “intermittency”.

2.2 Scaling Structure Functions and Special Cases

Define the increment $\Delta\phi(r; x) = \phi(x) - \phi(x - r)$ and consider the scaling of the structure functions:

$$\langle |\Delta\phi(r; x)|^p \rangle \sim r^{\zeta_\phi(p)} (\ell \leq r \leq L), \quad -1 < p < \infty \quad (3)$$

where normalization requires that $\zeta_\phi(0) = 0$. For the lower limit on p and ways to elevate it, see Ref. 7. Within the framework of turbulence (revisited below), $\zeta_\phi(p)$ is given a formal multifractal interpretation in Ref. 8 by using its Legendre transform. The associated $D_\phi(h)$ “spectrum” is understood to be the fractal dimension of the subset of $[0, L]$ where the local Hölder exponent $h(x) = h$, this last exponent being defined in $|\Delta\phi(r; x)| \sim r^{h(x)}$.

Weakly variable increments ($\langle |\Delta\phi(r; x)|^p \rangle \approx \langle |\Delta\phi(r; x)| \rangle^p$) lead directly to $\zeta_\phi(p) = pH$, in other words, simple- or mono-scaling structure functions where $0 < H < 1$. The former ($H \rightarrow 0$) limit brings us back to stationary scaling processes. The latter ($H \rightarrow 1$) limit corresponds to almost everywhere differentiable processes; in particular, this case contains noiseless trends: $\phi(x) = ax + b$ where a and b are random variables (as long as $a \neq 0$, almost surely). If $\zeta_\phi(p)$ is not linear then it is necessarily concave ($\zeta_\phi''(p) < 0$) and $H_\phi(p) = \zeta_\phi(p)/p$ will be a decreasing hierarchy of exponents. Two values of p have received a lot of attention.

Firstly, there exists a Fourier duality for non-stationary processes between $E_\phi(k)$ and $\langle |\phi(x) - \phi(x - r)|^2 \rangle$, often referred to as *the* structure function. For scaling processes, this leads to⁹:

$$\beta_\phi = 2H_\phi(2) + 1 \geq 1, \quad (4)$$

where the “=” is obtained in the stationary limit ($H_\phi(2) \rightarrow 0$). Beyond the opposite limit ($H_\phi(2) \rightarrow 1$), we find once differentiable functions ($\beta_\phi \geq 3$), hence $1 < \beta_\phi < 3$.

Secondly, one can relate the fractal dimension $D_{g(\phi)}$ of the graph $g(\phi)$ of $\phi(x)$, viewed as a self-affine¹⁰ object in 2-space, to

$$H_1 = H_\phi(1) = 2 - D_{g(\phi)} \geq 0. \quad (5)$$

The “=” is obtained again in the stationary ($H_1 \rightarrow 0$) limit where $D_{g(\phi)} = 2$ (the graph fills space). In the opposite ($H_1 \rightarrow 1$) limit of almost everywhere differentiable processes, we retrieve graphs with $D_{g(\phi)} = 1$, akin to rectifiable curves. In other words, stationarity comes with more “roughness” and discontinuity, non-stationarity with more “smoothness” and continuity in scaling processes.

In contrast to the above discussed case of narrowly distributed increments ($H_\phi(p) \equiv H$), we can talk about “multi-affine” processes¹ in the more interesting cases where $H_\phi(p)$ is non-constant. We will retain H_1 (quite literally) as a first order quantifier of scale-invariant non-stationarity and the whole $H_\phi(p)$ hierarchy as a means to qualify it. We provide elsewhere an illustration of these ideas using “bounded” cascade models [Ref. 11 and references therein].

3. SINGULARITY ANALYSIS, QUANTIFYING AND QUALIFYING INTERMITTENCY

3.1 Defining a Related Measure

We wish to perform a singularity analysis of the now relatively standard type (based on a measure) and not like in the above (Hölderian) variant. We need to define a non-negative field related to $\phi(x)$ which should furthermore be stationary, having a well-behaved auto-correlation function and a spectral exponent which does not exceed one. A convenient choice is provided by:

$$\delta\phi(x) = |\Delta\phi(\ell; x)| = |\phi(x) - \phi(x - \ell)|, \quad \ell \leq x \leq L, \quad (6)$$

which can be described as the “absolute (small-scale) gradient field”. Other options have been used.^{12–14}

The next step is to degrade the resolution of the field defined in Ref. 6. Namely, we compute:

$$p_\phi(r; x) = \sum_{x'/\ell=x/\ell-r/\ell+1}^{x'/\ell=x/\ell} \delta\phi(x'), \quad r \leq x \leq L(\ell \leq r \leq L), \quad (7)$$

and let

$$\varepsilon_\phi(r; x) = \frac{p_\phi(r; x)}{r}. \quad (8)$$

3.2 Scaling Properties and Special Cases

Consider the scalings of $\langle \varepsilon_\phi(r; x)^q \rangle$ as parameterized in Ref. 15:

$$\langle \varepsilon_\phi(r; x)^q \rangle \sim r^{-K_\varepsilon(q)} (\ell \leq r \leq L), \quad -\infty < q < \infty. \quad (9)$$

The exponent function $K_\varepsilon(q)$ is convex with predetermined values $K_\varepsilon(0) = K_\varepsilon(1) = 0$ and weak variability, $\langle p_\phi(r; x)^q \rangle \approx \langle p_\phi(r; x) \rangle^q$, yielding simply $K_\varepsilon(q) \equiv 0$. The multifractal significance of $K_\varepsilon(q)$ was established in Refs. 15 and 16 by showing that its Legendre transform yields the well-known singularity spectrum, $f_\varepsilon(\alpha)$.

Using the above parameterization, one can define the non-increasing hierarchy $D_\varepsilon(q) = 1 - K_\varepsilon(q)/(q - 1)$ known as generalized or Reyni dimensions.^{17,18} The single most important exponent in this whole approach is quite possibly:

$$C_1 = 1 - D_\varepsilon(1) = K'_\varepsilon(1) \geq 0, \quad (10)$$

the “information” codimension. It provides us with a straightforward measure of intermittency in the system, as determined by the deviation of $D_\epsilon(q)$ or $K_\epsilon(q)$ from a constant. The specific kind of intermittency is of course determined by the whole hierarchy $D_\epsilon(q)$.

4. GEOPHYSICAL APPLICATIONS

4.1 Effective Constitutive Laws in Scale-invariant Format

It is legitimate to ask whether or not both multifractal statistics are independent of each other. Do $\zeta_\phi(p)$ and $K_\epsilon(q)$ not convey, to some extent, the same information? We are convinced that the answer is yes, that the system is “physically” multifractal in a unique way. However, it is also clear that the two approaches capture only one facet each of this deep multifractal reality and that they are best viewed as complementary, at least in absence of any other knowledge.

Most geophysical fields of interest can be related to some globally conserved quantity (e.g. velocity and total kinetic energy). Any macroscopic (non-local) connection between a field $\phi(x)$ hence $\zeta_\phi(p)$, on the one hand, and its gradients $\delta\phi(x)$ hence $K_\epsilon(q)$, on the other hand, is bound to be related to the transport of this quantity, from place-to-place and/or scale-to-scale.

4.2 The Case of Fully Developed Turbulence

One instance where an extra *a priori* scaling relation is available — largely thanks to dimensional analysis — is 3D turbulence. In this problem, we can take $\phi(x) = u(x = |\mathbf{u}|t)$ using Taylor’s frozen turbulence hypothesis in a flow with mean velocity $\langle \mathbf{u} \rangle$. Indeed, a deterministic (event-per-event) interpretation of Kolmogorov’s famous (effective constitutive) relation²

$$\varepsilon_u(r; x) \approx \frac{|\Delta u(r; x)|^3}{r}, \quad r \leq x \leq L (\ell \leq r \leq L), \quad (11)$$

leads to

$$\zeta_u(p) = \frac{p}{3} - K_\epsilon \left(q = \frac{p}{3} \right). \quad (12)$$

Note that $\zeta_u(0) = 0$ as required and that $\zeta_u(3) = 1$ as expected from Eq. (11). The $K_\epsilon(p/3)$ term in Eq. (12) is the “intermittency” correction to the Kolmogorov’s² theory for homogeneous turbulence ($H_u(p) \equiv 1/3$). Recently, Eq. (11) has been the subject of some debate (Ref. 19 and references therein) and the consensus seems to be that it is best interpreted in a statistical sense, that it applies extremely well on average but can fail completely for any given event.

Returning to the case of arbitrary geophysical data, we can turn the above questions around. Given enough high-quality data, can one extract from it such powerful relations as Eq. (12) or even Eq. (11) using scale-invariance analysis? To this effect, we suggest a merger of the two basic (uni-variate) multifractal statistics into a single bi-variate one. Consider the scaling of:

$$\langle |\Delta\phi(r; x)|^p \varepsilon_\phi(r; x)^q \rangle \sim r^{X_\phi(p,q)} (\ell \leq r \leq L), \quad -1 < p < \infty, \quad -\infty < q < \infty. \quad (13)$$

In the above, we have looked at $X_\phi(p, 0) = \zeta_\phi(p)$ and $X_\phi(0, q) = -K_\epsilon(q)$. It is not hard to see that, in the unlikely situation where $|\Delta\phi(r; x)|$ and $\epsilon_{\Delta\phi}(r; x)$ are independent random variables, we have $X_\phi(p, q) = \zeta_\phi(p) - K_\epsilon(q)$. By way of contrast, the event-wise connection in Eq. (11) implies $X_u(p, q) = p/3 - K_\epsilon(q + p/3)$ which in turn yields Eq. (12). So, in principle, a careful bi-multifractal analysis of data can lead to the determination of important scaling relations.

5. DISCUSSION AND CONCLUSIONS

In summary, we surveyed two multifractal data analysis techniques based on structure functions and singular measures, yielding respectively the exponent hierarchies $H(q)$ and

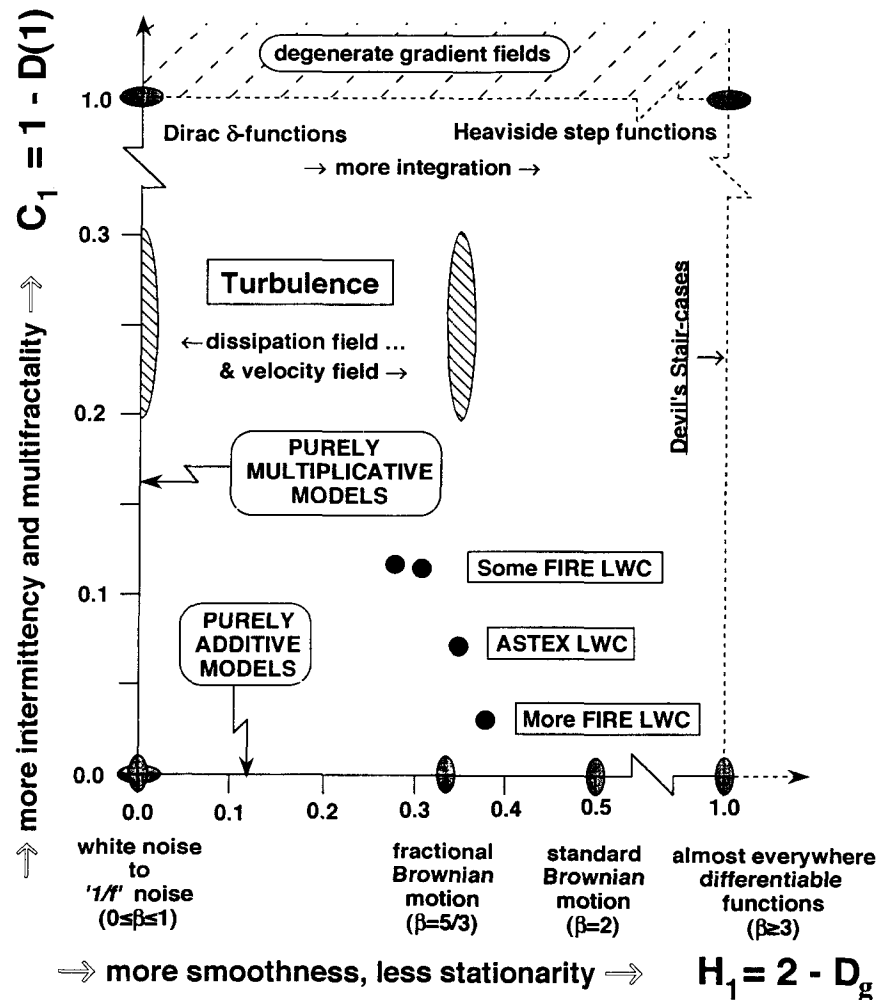


Fig. 1 The “ $q = 1$ multifractal plane”. Notice the natural boundaries of the accessible domain, $0 \leq H_1 \leq 1$, $0 \leq C_1 \leq 1$. This last limit prevents “degeneracy”,¹⁵ i.e., vanishing gradient fields in almost every realization (now-and-then a huge spike appears to keep the average at one); see text for the others. Several well-known models are found on these boundaries: “multiplicative” or turbulent cascade models, e.g. Ref. 9, “additive” models such as fractional Brownian motion and “Devil’s stair-cases”.¹⁰ In contrast, typical geophysical signals tend to live inside this domain, as illustrated by four cloud liquid water content (LWC) datasets obtained during two different campaigns (FIRE, ASTEX). Inside we would also find the handful of multi-affine models found in the literature [Ref. 11 and references therein].

$D(q)$. The latter is now a standard tool for quantifying and qualifying “intermittency” and we argued that the former (multi-affine) technique can be used analogously for the presently more fuzzy notion of “non-stationary”. We also merged the two approaches into a unified bi-multifractal one. Guiding ourselves with the well-studied case of 3D turbulence, we showed how this new concept can be used in a systematic quest for scale-invariant “effective constitutive laws.” It is hoped that, from these, insight can be gained into the nonlinear physical mechanisms at work generating, in particular, the geophysical signal being processed.

In the meantime, we show in Fig. 1 some of the model- and data-based activity happening on the “ $q = 1$ multifractal plane” where the coordinates are simply $H_1 = H(1)$ and $C_1 = 1 - D(1)$. In our view, the axes measure directly the degrees of non-stationarity (horizontally, H_1) and of intermittency (vertically, C_1) in the system. Non-vanishing intermittency leads to multi-scaling in singular measures (as a matter of definition) and also in structure functions (but in a non-trivial way). At this level of approximation, the rest — what precise flavor of multifractality is present — is a matter of detail. We advocate the

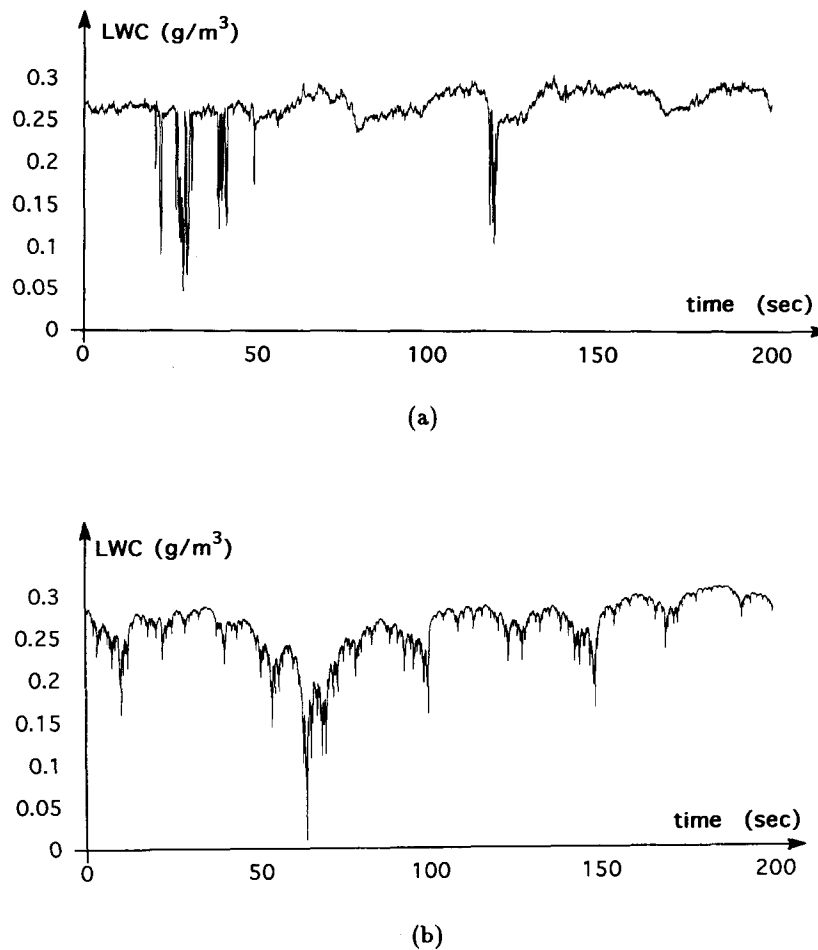


Fig. 2 (a) A sample of atmospheric liquid water density captured in a stratocumulus cloud deck. (b) A simple multi-affine stochastic model for the above, with comparable H_1 and C_1 . More specifically, we generate a fractionally integrated standard cascade model,¹⁵ and a linear transformation is then applied in order to fit the 1-point mean, variance and sign of skewness. This last procedure does not affect any of the exponents considered here since they are based on increments or on small scale differences.

use of such (H_1, C_1) -plots as simple universal tools in geophysical data analysis, whether it be for a classification study when several datasets are involved or as a way to determine the parameters of a stochastic model for a single dataset.²⁰ In Figs. 2(a) and 2(b) we illustrate this last exercise in the case of cloud structure. Figure 2(a) shows a typical *in situ* liquid water density probing described in more detail elsewhere²¹ and in Fig. 2(b) we see a multi-affine two-parameter model.^{15,20} We are confident that many other applications will be found.

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